Lec 38: 11/25/2009

Rotation in Two Dimensions (Cont'd):

The eigenstates of $\Phi_m(\phi)$ are simply $e^{im\phi}$. Note, however, that this must be a periodic function with period 2π. Thus:

$m = 0, \pm 1, \pm 2, \ldots$

$\Phi_m(\phi)$ is just a phase factor. We can then find $R_{Em}(\rho)$ for various values of m. What happens is that the equation for $R_{Em}(\rho)$ leads to a relationship between E and m (from boundary conditions, matching, etc).

Note that the eigenvalues of L_z can be used to uniquely specify all energy eigenstates with energy E. As an example, for a two-dimensional isotropic oscillator we find:

$E_n = (n+1)\hbar\omega \qquad n = 2k + |m|$

Here k is an integer and m is the L_z eigenvalue.

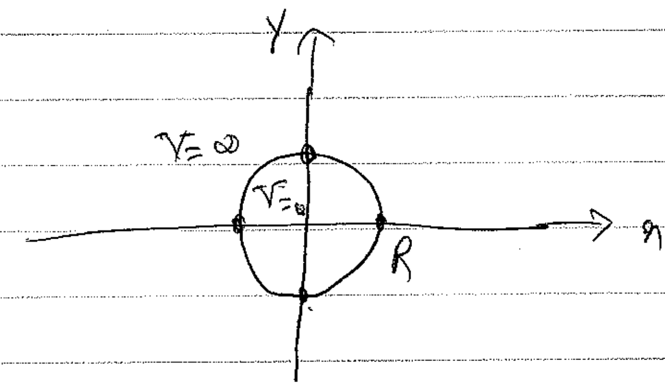
For a given "n" (hence E_n), "m" varies between "n" and 1 (if "n" is odd) or 0 (if "n" is even).

Note that "m" changes by units of two. The reason being that Ψ_{Em} is multiplied by a factor $(-1)^n$ under parity $(x \to -x, y \to -y)$. Under parity we have:

$$\rho \to \rho, \quad \phi \to \pi + \phi \qquad e^{im\phi} \to e^{im\pi} e^{im\phi}$$

Since all eigenstates with a given energy (given "n") have the same parity, then $e^{im\pi}$ must be +1 or −1 for all of these states. Therefore "m" must be even (if "n" is even) or odd (if "n" is odd).

Another example is a particle in a circular box of radius R:



The equation for the radial **part** of the wavefunction is;

$$\frac{-\hbar^2}{2\mu}\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}\right]R_{Em}(\rho)=E\,R_{Em}(\rho)$$

The solutions are Bessel functions and the boundary condition is $R_{Em}(\rho)=0$.

Rotation in Three Dimensions:

The angular momentum is a vector in three dimensions;

$$\vec{L}=\vec{R}\times\vec{P}$$

$$L_x=YP_z-ZP_y\ ,\ L_y=ZP_x-XP_z\ ,\ L_z=XP_y-YP_x$$

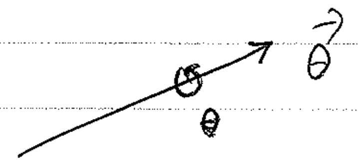
The components of $\vec{L}$ satisfy the following commutation relations:

$$[L_x,L_y]=i\hbar L_z\ ,\ [L_y,L_z]=i\hbar L_x\ ,\ [L_z,L_x]=i\hbar L_y$$

The total angular momentum operator $L=\sqrt{L_x^2+L_y^2+L_z^2}$ commutes with all the three components.

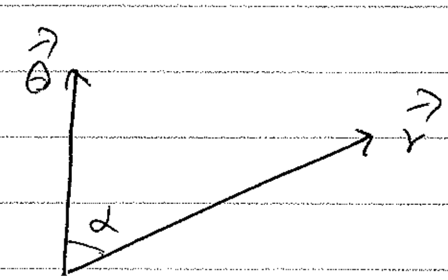
$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$

L_x, L_y, L_z are generators of rotation about the x, y, z axes respectively. A general rotation is specified by a vector $\vec{\theta}$, the direction gives the direction of the axis about which rotation happens (using the right hand rule) and the magnitude is the angle of rotation.



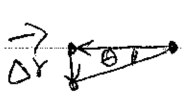
Under a general rotation by angle θ ($\theta \ll 1$), a point $\vec{r} = (x, y, z)$ is transformed to $\vec{r} + \vec{\theta} \times \vec{r}$:

$\times$ → cross product



Side view

Top view



$\vec{\Delta r} = \vec{\theta} \times \vec{r} = r\theta \sin\alpha \; \hat{\theta} \times \hat{r}$ → unit vectors

An arbitrary state vector $|\Psi\rangle$ is then transformed as;

$$\langle x, y, z | \Psi\rangle \to \langle x, y, z| R(\vec{\theta}) |\Psi\rangle = \langle \vec{r} - \vec{\theta} \times \vec{r} | \Psi\rangle$$

Similar to what we did for rotation in two dimensions we find;

$$R(\vec{\theta}) = \exp\left(\frac{-i}{\hbar} \vec{\theta} \cdot \vec{L}\right) = \exp\left[-\frac{i}{\hbar}(\theta_x L_x + \theta_y L_y + \theta_z L_z)\right]$$

Note that this is not the same as rotating about the x-axis by angle θ_x, then rotating about the y-axis by angle θ_y, and finally rotating about the z-axis by angle θ_z. Rotations about the x, y, z axes do not commute (a direct consequence of the fact that L_x, L_y, L_z do not commute).

Therefore $\theta_x, \theta_y, \theta_z$ should be interpreted as the components of the vector $\vec{\theta}$ only.

Rotational Invariance in Three Dimensions:

A system is rotationally invariant in three dimensions if its Hamiltonian remains invariant under an arbitrary rotation. This means that:

$$[H, L_x] = [H, L_y] = [H, L_z] = 0 \Rightarrow [H, L^2] = 0$$

Out of the five operators H, L_x, L_y, L_z, L^2 there is a maximum number of three commuting operators:

$$[H, L_z] = [H, L^2] = [L_z, L^2] = 0$$

For a rotationally invariant system, H has the same eigenstates as L_z and L^2. Thus the eigenvalues of L_z and L^2 can be used to uniquely label degenerate energy eigenstates.